Generalized robust toric ideals

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A R T I C L E   I N F O

A B S T R A C T

An ideal I is robust if its universal Gröbner basis is a minimal generating set for this ideal. In this paper, we generalize the meaning of robust ideals. An ideal is defined as generalized robust if its universal Gröbner basis is equal to its universal Markov basis. This article consists of two parts. In the first one, we study the generalized robustness on toric ideals of a graph G. We prove that a toric graph ideal is generalized robust if and only if its universal Markov basis is equal to the Graver basis of the ideal. Furthermore, we give a graph theoretical characterization of generalized robust graph ideals, which is based on terms of graph theoretical properties of the circuits of the graph G. In the second part, we go on to describe the general case of toric ideals, in which we prove that a robust toric ideal has a unique minimal system of generators, or in other words, all of its minimal generators are indispensable.

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1. Introduction

Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subseteq \mathbb{N}^n$ be a vector configuration in $\mathbb{Q}^n$ and $\mathbb{N}A := \{l_1\mathbf{a}_1 + \cdots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}\}$ the corresponding affine semigroup, where $\mathbb{N}A$ is pointed, that is, if $x \in \mathbb{N}A$ and $-x \in \mathbb{N}A$ then $x = \mathbf{0}$. We grade the polynomial ring $\mathbb{K}[x_1, \ldots, x_m]$ over an arbitrary field $\mathbb{K}$ by the semigroup $\mathbb{N}A$ setting $\deg_A(x_i) = \mathbf{a}_i$ for $i = 1, \ldots, m$. For $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$, we define the A-degree of the monomial $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \cdots x_m^{u_m}$ to be

$$\deg_A(\mathbf{x}^{\mathbf{u}}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A.$$

The toric ideal I_A associated with A is the prime ideal generated by all the binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ such that $\deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}})$, see [21].

Toric ideals constitute a special class of ideals in a polynomial ring. They define toric varieties, a large class of algebraic varieties, that play an important role in the development of mathematics the last years. Their study starts with Hochster in [14] and spreads through a series of lectures by Fulton, see [11,12].

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As far as the applicability of toric ideals is concerned, it has to be mentioned that toric ideals are related to recent advances in polyhedral geometry, toric geometry, algebraic geometry, algebraic statistic, integer programming, graph theory, computational algebra, etc., where they are applied in a natural way, see for example [9,10,17,21].

There are several sets for a toric ideal, which include crucial information about it, such as the Graver basis, the universal Markov basis, the universal Gröbner basis and the set of the circuits. An irreducible binomial $\mathbf{x^u} - \mathbf{x^v}$ in I_A is called primitive if there is no other binomial $\mathbf{x^w} - \mathbf{x^z}$ in I_A, such that $\mathbf{x^w}$ divides $\mathbf{x^u}$ and $\mathbf{x^z}$ divides $\mathbf{x^v}$. The set of the primitive binomials forms the Graver basis of I_A and is denoted by Gr_A. As it is known by a theorem of Diaconis and Sturmfels, every minimal generating set of I_A corresponds to a minimal Markov basis of A, which is denoted by M_A, see [9, Theorem 3.1]. The universal Markov basis of A is denoted by $\mathcal{M}_A$ and is defined as the union of all minimal Markov bases of A, see [15, Definition 3.1]. The universal Gröbner basis of an ideal I_A, which is denoted by $\mathcal{U}_A$, is a finite subset of I_A and it is a Gröbner basis for the ideal with respect to all admissible term orders, see [21]. The support of a monomial $\mathbf{x^u}$ of $\mathbb{K}[x_1, \ldots, x_m]$ is $\mathrm{supp}(\mathbf{x^u}) := \{i \mid x_i \text{ divides } \mathbf{x^u}\}$ and the support of a binomial $B = \mathbf{x^u} - \mathbf{x^v}$ is $\mathrm{supp}(B) := \mathrm{supp}(\mathbf{x^u}) \cup \mathrm{supp}(\mathbf{x^v})$. An irreducible binomial is called circuit if it has minimal support. The set of the circuits of a toric ideal I_A is denoted by $\mathcal{C}_A$. The relation between the above sets was studied by B. Sturmfels in [21]:

Proposition 1.1. *(See [21, Proposition 4.11].) For any toric ideal I_A it holds:*

$$\mathcal{C}_A \subseteq \mathcal{U}_A \subseteq Gr_A.$$

An ideal I is called robust if its universal Gröbner basis is equal to a Markov basis of the ideal. Robustness is a property of ideals that has not been fully described. More specifically, it has been described for toric ideals which are generated by quadratics. Toric ideals which are generated by quadratics were studied by Ohsugi and Hibi in [18], while the robustness for this class of ideals is described in the article of Boocher and Robeva, see [2]. The importance of robustness stems from the interest in the study of ideals which are minimally generated by a Gröbner basis for an arbitrary term order, see [8]. Moreover, the study of robustness is important, due to the fact that several areas of mathematics are keen on the research of the Markov basis, the universal Gröbner basis and the Graver basis of an ideal. This problem has also been researched in the case of toric ideals arising from a graph G, as studied by Boocher et al. in [3]. In their work the authors proved that any robust toric ideal of a graph G is also minimally generated by its Graver basis, [3, Theorem 3.2]. In addition, they completely characterize all graphs which give rise to robust ideals, see [3, Theorem 4.8].

The present article generalizes the meaning of robust ideals. A robust ideal is called generalized robust if its universal Gröbner basis is equal to its universal Markov basis. This manuscript is divided into two parts.

In the first part, we study the generalized robustness on toric ideals of a graph G. The results of this part are inspired and guided by the work of [3] in order to give theorems that fully characterize the generalized robust toric ideals of graphs. The papers [20,22,24] describe the Markov basis, the Graver basis, the universal Gröbner basis and the set of the circuits for a toric ideal arising from a graph. In Section 2, we analyze all these notions more explicitly. Applying this knowledge on the work of Boocher et al. (see [3]), we are allowed to provide the study of the generalized robustness of graphs, with theorems of the same structure as theirs. In Section 3, we first prove that a toric graph ideal is generalized robust if and only if its universal Markov basis is equal to the Graver basis of the ideal, see Theorem 3.4. Moreover, the relation between robust graph ideals and generalized robust graph ideals is studied. In the next section, we go on to give a graph theoretical characterization of generalized robust graph ideals, which is based on terms of graph theoretical properties of the circuits of the graph G, see Theorem 4.5.

In the second part of this manuscript, we study the robustness property in the general case of toric ideals. More specifically, we study the indispensable binomials which exist in a robust toric ideal. A binomial $B \in I_A$ is called indispensable if there exists a non-zero constant multiple of it in every minimal system of binomial generators of I_A. A recent problem arising from algebraic statistics is to find classes of toric ideals which have a unique minimal system of generators, see [1,4]. In order to study this problem, Ohsugi and Hibi introduced in [19] the notion of indispensable binomials. In Section 5, we prove that a robust toric ideal has a unique minimal system of generators, or in other words, all of its minimal generators are indispensable, see Theorem 5.10. Finally, we conclude that the robustness property for a toric ideal, implies the generalized robustness property for it, see Corollary 5.12. In conclusion, we present a family of toric ideals, for which $\mathcal{C}_A = Gr_A$, see Remark 5.14.

2. Elements of the toric ideals of graphs

In the next chapters, G is a connected, undirected, finite, simple graph on the vertex set $V(G) = \{v_1, \ldots, v_n\}$. Let $E(G) = \{e_1, \ldots, e_m\}$ be the set of edges of G and $\mathbb{K}[e_1, \ldots, e_m]$ be the polynomial ring in the m variables $e_1, \ldots, e_m$ over an arbitrary field $\mathbb{K}$. We will associate each edge $e = \{v_i, v_j\} \in E(G)$ with the element $a_e = v_i + v_j$ in the free abelian group $\mathbb{Z}^n$, with basis the set of the vertices of G, where $v_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the vector with 1 in the i-th coordinate of v_i. With I_G we denote the toric ideal I_{A_G} in $\mathbb{K}[e_1, \ldots, e_m]$, where $A_G = \{a_e \mid e \in E(G)\} \subset \mathbb{Z}^n$.

A walk of length q connecting $v_{i_1} \in V(G)$ and $v_{i_{q+1}} \in V(G)$ is a finite sequence of the form $w = (e_{i_1} = \{v_{i_1}, v_{i_2}\}, e_{i_2} = \{v_{i_2}, v_{i_3}\}, \ldots, e_{i_q} = \{v_{i_q}, v_{i_{q+1}}\})$ with each $e_{i_j} = \{v_{i_j}, v_{i_{j+1}}\} \in E(G)$. We call a walk $w' = (e_{j_1}, \ldots, e_{j_t})$ a subwalk of w if $e_{j_1} \cdots e_{j_t} | e_{i_1} \cdots e_{i_q}$. An even (respectively odd) walk is a walk of even (respectively odd) length. A walk as w is called closed if $v_{i_{q+1}} = v_{i_1}$. A cycle is a closed walk with $v_{i_k} \neq v_{i_j}$, for every $1 \leq k < j \leq q$. Depending on the property of the walk that we want to emphasize, we may denote a walk w either by a sequence of vertices and edges $(v_{i_1}, e_{i_1}, v_{i_2}, \ldots, v_{i_q}, e_{i_q}, v_{i_{q+1}})$ or exclusively with vertices $(v_{i_1}, v_{i_2}, v_{i_3}, \ldots, v_{i_{q+1}})$ or only with edges $(e_{i_1}, \ldots, e_{i_q})$. Note that, although the graph G has no multiple edges, since it is simple, the same edge e may appear more than once in a walk. In this case, e is called multiple edge of the walk w. Given an even closed walk $w = (e_{i_1}, e_{i_2}, \ldots, e_{i_{2q}})$ of the graph G, we denote by B_w the binomial

$$B_w = \prod_{k=1}^{q} e_{i_{2k-1}} - \prod_{k=1}^{q} e_{i_{2k}}$$

belonging to the toric ideal I_G. Actually, the toric ideal I_G is generated by binomials of the above form, see [24].

For convenience, by $\mathbf{w}$ we denote the subgraph of G, whose vertices and edges are the vertices and the edges of the walk w. Note that $\mathbf{w}$ is a connected subgraph of G. A cut edge (respectively a cut vertex) is an edge (respectively a vertex) of the graph, whose removal increases the number of connected components of the remaining subgraph. A graph is called biconnected if it is connected and does not contain a cut vertex. A block $\mathcal{B}$ is a maximal biconnected subgraph of a given graph G. That means any other subgraph of G that strictly contains $\mathcal{B}$ is disconnected or has a cut vertex.

A walk w of a graph is primitive if and only if the corresponding binomial B_w is primitive. Every even primitive walk $w = (e_{i_1}, \ldots, e_{i_{2k}})$ partitions the set of its edges in two sets $w^+ = \{e_{i_j} \mid j \text{ odd}\}$ and $w^- = \{e_{i_j} \mid j \text{ even}\}$, otherwise the binomial B_w is not irreducible. The edges of the set w^+ are called odd edges of the walk and those of w^- even. A sink of a block $\mathcal{B}$ is a common vertex of two odd or two even edges of the walk w which belong to the block $\mathcal{B}$. Finally, we call strongly primitive walk, a primitive walk which does not have two sinks with distance one in any cyclic block of the walk, or equivalently does not have two adjacent cut vertices in any cyclic block of w.

In [20] a complete characterization of the Graver basis of the corresponding toric ideal I_G was given, see [20, Theorem 3.2]. The next corollary, given by the same authors, describes the structure of the underlying graph of a primitive walk.

Corollary 2.1. *(See [20, Corollary 3.3].) Let G be a graph and W a connected subgraph of G. The subgraph W is the graph $\mathbf{w}$ of a primitive walk w if and only if*

(1) W is an even cycle or
(2) W is not biconnected and
    (a) every block of W is a cycle or a cut edge and
    (b) every cut vertex of W belongs to exactly two blocks and separates the graph in two parts, the total number of edges of the cyclic blocks in each part is odd.

Afterwards, we recall from [20], a lot of graph theoretical notions in order to describe the universal Markov basis of a toric ideal of a graph G. A binomial is called minimal, if it belongs to at least one minimal system of generators of I_G, i.e. at least one Markov basis of I_G.

For a given subgraph F of G, an edge f of the graph G is called chord of the subgraph F, if the vertices of the edge f belong to $V(F)$ and $f \notin E(F)$. A chord $e = \{v_k, v_l\}$ is called bridge of a primitive walk w if there exist two different blocks $\mathcal{B}_1, \mathcal{B}_2$ of $\mathbf{w}$ such that $v_k \in \mathcal{B}_1$ and $v_l \in \mathcal{B}_2$. Let w be an even closed walk $(\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{2q}, v_1\})$ and $f = \{v_i, v_j\}$ a chord of w. Then, f breaks w into two walks:

$$w_1 = (e_1, \dots, e_{i-1}, f, e_j, \dots, e_{2q})$$

and

$$w_2 = (e_i, \dots, e_{j-1}, f),$$

where $e_s = \{v_s, v_{s+1}\}$, $1 \leq s \leq 2q$ and $e_{2q} = \{v_{2q}, v_1\}$. The two walks are both even or both odd. A chord is called even (respectively odd) if it is not a bridge and if it breaks the walk into two even walks (respectively odd).

Let $w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \cdots, \{v_{i_{2q}}, v_{i_1}\})$ be a primitive walk. Let $f = \{v_{i_s}, v_{i_j}\}$ and $f' = \{v_{i_{s'}}, v_{i_{j'}}\}$ be two odd chords (that is, they are not bridges and the numbers $j - s$, $j' - s'$ are even) with $1 \leq s < j \leq 2q$ and $1 \leq s' < j' \leq 2q$. We say that f and f' cross effectively in w if $s' - s$ is odd (then necessarily $j - s'$, $j' - j$, $j' - s$ are odd) and either $s < s' < j < j'$ or $s' < s < j' < j$. We call F_4 of the walk w, a cycle (e, f, e', f') of length four which consists of two edges e, e' of the walk w both odd or both even, and two odd chords f, f' which cross effectively in w. An F_4, (e_1, f_1, e_2, f_2), separates the vertices of $\mathbf{w}$ into two parts $V(\mathbf{w}_1)$, $V(\mathbf{w}_2)$, since both edges e_1, e_2 of the F_4 belong to the same block of $w = (w_1, e_1, w_2, e_2)$. We say that an odd chord f of a primitive walk $w = (w_1, e_1, w_2, e_2)$ crosses an F_4, (e_1, f_1, e_2, f_2), if one of the vertices of f belongs to $V(\mathbf{w}_1)$, the other in $V(\mathbf{w}_2)$ and f is different from f_1 and f_2.

The next theorem by Reyes et al., from which we know the elements of the universal Markov basis of the ideal I_G, gives a necessary and sufficient characterization of the minimal binomials of a toric ideal of a graph G.

Theorem 2.2. *(See [20, Theorem 4.13].) Let w be an even closed walk. B_w is a minimal binomial if and only if*

(M1) all the chords of w are odd,
(M2) there are not two odd chords of w which cross effectively except if they form an F_4,

(M3) no odd chord crosses an F_4 of the walk w,
(M4) w is a strongly primitive.

3. Generalized robust toric ideals of graphs

In this section we study the generalized robust toric ideals of graphs.

Definition 3.1. An ideal I is called generalized robust if its universal Gröbner basis is equal to its universal Markov basis.

Undoubtedly, it is a hard problem to characterize the generalized robustness, owing to the fact that only for a few classes of toric ideals we know their universal Gröbner basis and the universal Markov basis. In general, characterizing and computing these sets, is a difficult and computationally demanding problem. Lawrence ideals provide a large class of generalized robust toric ideals, since it is known by Sturmfels that in a Lawrence ideal any minimal generating set coincides with the universal Gröbner basis and the Graver basis, see [21, Theorem 7.1]. Moreover, not only the robustness but also the generalized robustness is not a property which describes completely the Lawrence ideals, see [2, Example 3.4].

In order to describe the universal Gröbner basis for the case of toric ideals of graphs, we give the notions of pure blocks and of the mixed walks of a graph G, see [22]. A cyclic block $\mathcal{B}$ of a primitive walk w is called pure if all the edges of the block $\mathcal{B}$ belong either to $\mathbf{w}^+$ or to $\mathbf{w}^-$. A primitive walk w is called mixed if none of the cyclic blocks of w is pure. The next theorem describes completely the elements of the universal Gröbner basis of a toric ideal of a graph G.

Theorem 3.2. *(See [22, Theorem 3.4].) Let w be a primitive walk. B_w belongs to the universal Gröbner basis of I_G if and only if w is mixed.*

Based on the above theorem, in combination with the knowledge of the universal Markov basis for a toric ideal of a graph G, we are allowed to research in depth the generalized robust graph ideals.

In the special case of toric ideals of graphs, a useful property for every minimal generator of the ideal, is that it belongs to its universal Gröbner basis, as we can see in the next proposition.

Proposition 3.3. *Let G be a graph and I_G its corresponding toric ideal. Then*

$$\mathcal{M}_G \subseteq \mathcal{U}_G.$$

Proof. Let G be a graph and I_G its corresponding toric ideal. Let B_w be an element of the universal Markov basis of I_G, which means by definition that B_w is a minimal generator of the ideal. We will prove that the binomial B_w belongs to the universal Gröbner basis of I_G.

We assume that B_w does not belong to the $\mathcal{U}_G$. By Theorem 3.2 the walk w is not mixed. Therefore, the walk w has at least one pure cyclic block and let it be $\mathcal{B} = (e_1, \ldots, e_n)$. Thus, all the edges e_i of the block $\mathcal{B}$ are either even or odd. Let $e_1 = (u_1, u_2)$, $e_2 = (u_2, u_3)$ and $e_3 = (u_3, u_4)$ be three consecutive edges of the block $\mathcal{B}$. We know that the block $\mathcal{B}$ has at least three edges, since $\mathcal{B}$ is a cycle of the graph G. The vertices u_2 and u_3 are both common vertices of either two odd or two even edges of $\mathcal{B}$. In consequence, the vertices u_2 and u_3 are sinks of the walk w with distance one. We remark that w is primitive, since B_w is a minimal generator of the ideal. It follows that w is not strongly primitive, a contradiction arises due to the minimality of B_w and Theorem 2.2. $\quad\square$

We mention that the above argument is not true in the general case of toric ideals. In [5] the authors provide a counterexample for this claim, [5, Example 1.8].

We are now ready to prove our main result in this section, in which we describe the generalized robustness for graph ideals. In [3] the authors proved that a graph ideal is robust if and only if the Graver basis of the ideal is equal to a Markov basis of it. Next, we are proving the corresponding theorem for a generalized robust ideal which is stated on its Graver basis and its universal Markov basis of the ideal.

Theorem 3.4. *Let G be a graph and I_G its corresponding ideal. The ideal I_G is generalized robust if and only if $\mathcal{M}_G = Gr_G$.*

Proof. Let G be a graph and I_G its corresponding toric ideal. By Proposition 1.1 and Proposition 3.3 we have that

$$\mathcal{M}_G \subseteq \mathcal{U}_G \subseteq Gr_G.$$

If $\mathcal{M}_G = Gr_G$, then the result follows.

Conversely, let I_G be a generalized robust ideal, which means that $\mathcal{M}_G = \mathcal{U}_G$. It is enough to prove that $Gr_G \subseteq \mathcal{U}_G$. Let w be a walk of the graph G such that $B_w \in Gr_G$. We will prove that $B_w \in \mathcal{U}_G$. Suppose this is not true. Therefore, the walk w has at least one pure cyclic block, and let it be $\mathcal{B} = (e_1, \ldots, e_n)$. The walk w has the form $w = (w_1, e_1, w_2, e_2, \ldots, e_{n-1}, w_n, e_n)$, where $w_1, \ldots, w_n$ are subwalks of w, as in Fig. 1. We remark that the walks $w_1, \ldots, w_n$ are odd. The reason is that the walk w is primitive, which means that every cut vertex of w separates the graph into two parts, the total number of edges of the cyclic blocks in each part is odd, see Corollary 2.1.
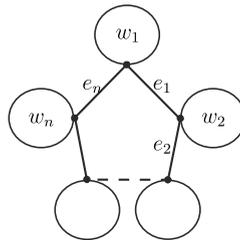

Fig. 1. The walk w.

Furthermore, by Corollary 2.1 we know that every block of w is a cycle or a cut edge. Therefore, we can assume that each one of the walks w_1 and w_n has at least one odd cycle. Let them be c_1 and c_n correspondingly. We will prove the existence of a walk q of the graph G, such that B_q belongs to $\mathcal{U}_G$ but not to $\mathcal{M}_G$.

We consider the walk

$$q = (c_1, p_1, e_1, e_2, \ldots, e_{n-1}, p_2, c_n, -p_2, e_{n-1}, \ldots, e_1, p_1)$$

which consists of the two odd cycles c_1, c_n and the path $p = (p_1, e_1, \ldots, e_{n-1}, p_2)$ joining them, where p_1, p_2 are paths of w which join the cycles c_1 and c_n with the edges e_1 and e_{n-1} correspondingly. By Theorem 3.2 it follows that the binomial B_q is an element of the $\mathcal{U}_G$. We remark that the edge e_n of w is a bridge of the walk q. By Theorem 2.2 it follows that the binomial B_q is not minimal. As a result, the binomial B_q does not belong to the universal Markov basis of the ideal I_G, a contradiction arises. $\square$

From the above theorem it follows that the robustness implies the generalized robustness for a toric graph ideal.

Corollary 3.5. *Let I_G be a robust ideal of a graph G. The ideal I_G is generalized robust.*

The converse of the above corollary is not true as the following example proves.

Example 3.6. We consider the complete graph $G = K_4$ (see Fig. 2) on the four vertices and let I_G be its corresponding toric ideal.
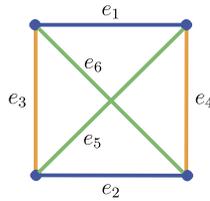

Fig. 2. A generalized robust ideal which is not robust.

The ideal has three minimal generators, which are

$$B_{w_1} = e_1e_2 - e_5e_6, \ B_{w_2} = e_3e_4 - e_5e_6 \text{ and } B_{w_3} = e_1e_2 - e_3e_4.$$

Moreover, it has three minimal systems of generators

$$M_1 = \langle B_{w_1}, B_{w_2} \rangle, \ M_2 = \langle B_{w_1}, B_{w_3} \rangle \text{ and } M_3 = \langle B_{w_2}, B_{w_3} \rangle,$$

and as a result its universal Markov basis is $\mathcal{M}_G = \langle B_{w_1}, B_{w_2}, B_{w_3} \rangle$. Since the walks w_1, w_2, w_3 are even cycles and no other block exists on the graph G, by Corollary 2.1 it follows that there is no other primitive element of I_G and therefore $Gr_G = \langle B_{w_1}, B_{w_2}, B_{w_3} \rangle$. Thus, the ideal I_G is generalized robust but not robust.

In order to check the converse statement of the last corollary, we will use the following corollary, as it was presented in [20].

Corollary 3.7. *(See [20, Corollary 4.15].) Let G be a graph which has no cycles of length four. The toric ideal I_G has a unique system of binomial generators.*

Proposition 3.8. *Let I_G be a generalized robust ideal. If the graph G has no cycles of length four, then I_G is robust.*

Proof. Let I_G be a generalized robust ideal, where G has no cycles of length four. By definition we know that $\mathcal{M}_G = \mathcal{U}_G$. By Corollary 3.7 it follows that I_G has a unique system of minimal generators, which means that $M_G = \mathcal{M}_G$. Therefore, the ideal I_G is minimally generated by its Gröbner basis. It follows that I_G is robust. □

The converse of the above proposition is not true as the following example proves.

Example 3.9. Let G be the graph which is a chordless cycle of length four, see Fig. 3.

Then, the corresponding toric ideal I_G has one minimal generator the $B_w = \langle e_1e_2 - e_3e_4 \rangle$. Obviously, the ideal has a unique system of minimal generators, the $M_G = \langle B_w \rangle$ and therefore $\mathcal{M}_G = \langle B_w \rangle$. It is clear that the Graver basis of I_G consists of exactly the binomial B_w. It follows that the ideal is both robust and generalized robust, but it contains a cycle of length four.

Certainly, the uniqueness of the minimal system of generators of an ideal is a sufficient and a necessary condition for the toric ideal of a graph G, to be robust if it is generalized robust and conversely.
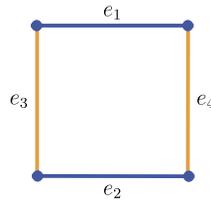
Fig. 3. A both robust and generalized robust graph.

4. Circuits and generalized robust graph ideals

In this section, we will present a graph theoretical characterization of a generalized robust toric ideal of a graph G, which is based on terms of graph theoretical properties of the circuits of the graph G.

The circuits of a graph G were described in graph theoretical terms with necessary and sufficient conditions by R. Villarreal:

Proposition 4.1. *(See [24, Proposition 4.2].) Let G be a finite connected graph. The binomial $B \in I_G$ is a circuit if and only if $B = B_w$ where*

(C1) w is an even cycle or
(C2) w consists of two odd cycles intersecting in exactly one vertex or
(C3) w consists of two vertex disjoint odd cycles joined by a path.

From [18] we also know the form of the primitive walks of a graph G.

Lemma 4.2. *(See [18, Lemma 3.2].) If B_w is primitive, then w has one of the following forms:*

(P1) w is an even cycle or
(P2) w consists of two odd cycles intersecting in exactly one vertex or
(P3) $w = (c_1, w_1, c_2, w_2)$ where c_1, c_2 are odd vertex disjoint cycles and w_1, w_2 are walks which combine a vertex v_1 of c_1 and a vertex v_2 of c_2.

The converse of the above lemma is also true, if a walk w has either the form (P1) or the form (P2). However, this is not true when the walk w has the form (P3). As we saw, Corollary 2.1 describes completely the primitive graphs. It is clear that in the case that the walk w is either of the form (P1) or of the form (P2), then it is always strongly primitive, since the corresponding primitive graph **w** does not contain two sinks.

The next proposition describes some properties of the primitive elements of a generalized robust graph ideal which will be our main tool, to prove the main result in this section.

Proposition 4.3. *The ideal I_G is generalized robust if and only if all its primitive elements satisfy the conditions (M1) and (M2) of Theorem 2.2.*

Proof. Let I_G be a generalized robust ideal. By Theorem 3.4 all primitive elements are minimal generators and therefore they satisfy the conditions (M1) and (M2). Conversely, we assume that all the primitive elements of I_G satisfy the conditions (M1) and (M2) of Theorem 2.2. In order to prove that the ideal I_G is generalized robust, by Theorem 3.4 and Proposition 3.3 we have to prove that $Gr_G \subseteq \mathcal{M}_G$. Let B_w be an element of Gr_G such that it satisfies the conditions (M1) and (M2) of Theorem 2.2. We have to prove that the binomial B_w is minimal. By hypothesis, it remains to prove that B_w satisfies the conditions

(M3) and (M4) of Theorem 2.2, i.e. the walk w does not have an odd chord which crosses an F_4 of w and the walk w is strongly primitive.

Firstly, we prove that the walk w does not have an odd chord which crosses an F_4 of w. Suppose not. So, there exists an odd chord $f = \{v_1, v_2\}$ that crosses the F_4, (e_1, f_1, e_2, f_2) of the walk $w = (w_1, e_1, w_2, e_2)$, see Fig. 4.
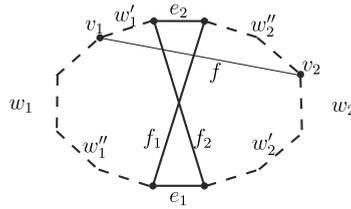

Fig. 4. An odd chord which crosses an F_4.

Then, w can be written in the form $(w_1', \{v_1\}, w_1'', e_1, w_2', \{v_2\}, w_2'', e_2)$, where $w_1 = (w_1', \{v_1\}, w_1'')$ and $w_2 = (w_2', \{v_2\}, w_2'')$. Since the chord f is odd, by definition the walks (f, w_2'', e_2, w_1') and (f, w_1'', e_1, w_2') are both odd. In addition, since (e_1, f_1, e_2, f_2) is an F_4, the walks w_1 and w_2 are both odd. Therefore, $(w_1'', f_1, -w_2'', f)$ and $(w_1', f, -w_2', f_2)$ are both even. Hence, by the definition, f is an even chord of $w' = (w_1, f_1, -w_2, f_2)$. Since the walk w is primitive, the walk w' is also primitive and w' has an even chord, a contradiction arises due to the fact that by hypothesis all primitive elements satisfy the condition (M1) of Theorem 2.2.

It remains to prove that w is strongly primitive. Suppose not. Therefore, the walk w has the form $w = (c_1, w_1, c_2, w_2)$, where c_1, c_2 are odd vertex disjoint cycles and w_1, w_2 are walks which combine a vertex v_1 of c_1 and a vertex v_2 of c_2. Since the walk is not strongly primitive, there is a cyclic block $\mathcal{B}$ of the primitive walk w in which there are two adjacent cut vertices u_1 and u_2 of w. Let the block be $\mathcal{B} = (q, e)$, where e is the edge $\{u_1, u_2\}$ and q be the path of the cyclic block $\mathcal{B}$ which connects the vertices u_1 and u_2. Then, w can be written in the form

$$w = (c_1, \{v_1\}, w', \{u_1\}, q, \{u_2\}, w'', \{v_2\}, c_2, \{v_2\}, -w'', \{u_2, u_1\} = e, -w'),$$

where $w_1 = (\{v_1\}, w', \{u_1\}, q, \{u_2\}, w'', \{v_2\})$ and $w_2 = (\{v_2\}, -w'', e, -w')$. Since the graph is connected, there is a path between any two vertices of the graph. Let p_1 be a path between the vertices v_1 and u_1 and let p_2 be a path between the vertices v_2 and u_2. Without loss of generality, we assume that the paths p_1 and p_2 are subwalks of the subwalks w' and w'' of the walk w. We consider the walk

$$w_c = (c_1, p_1, \{u_1\}, q, \{u_2\}, -p_2, c_2, p_2, \{u_2\}, -q, \{u_1\}, -p_1).$$

The above walk is a circuit since it is in the form (C3), where c_1, c_2 are the two vertex disjoint odd cycles and $p = (p_1, q, -p_2)$ the path which joins them. Therefore, the binomial B_{w_c} is primitive and the edge $e = \{u_1, u_2\}$ is a bridge of w_c. By hypothesis all the primitive elements of I_G satisfy the condition (M1) of Theorem 2.2, which means that they have no bridges. A contradiction arises. $\square$

In [3, Theorem 4.8] the authors proved the following lemma:

Lemma 4.4. *Let I_G be an ideal such that all the chords of all of its primitive elements are odd. Then, there is no circuit of G which shares exactly one edge (and no other vertices) with another circuit such that the shared edge is part of a cyclic block in both circuits.*

We are ready to present the main result of this section. The only difference between it and the corresponding result of Boocher et al. in [3], in the case of robust graph ideals, is that for an ideal I_G to be generalized robust we allow to the circuits of the graph the existence of two odd chords of the walk w which form an F_4. As a result of the previous proposition, the proof of the following theorem respects completely the construction of the corresponding proof of Theorem 4.8 in [3]. In this part of the proof we refer to the corresponding proof of them.

Theorem 4.5. *The ideal I_G is generalized robust if and only if the following conditions are satisfied.*

(R1) *No circuit of G has either an even chord or a bridge.*
(R2) *No circuit of G contains two odd chords which cross effectively, except if they form an F_4.*
(R3) *No circuit of G shares exactly one edge (and no other vertices) with another circuit such that the shared edge is a part of a cyclic block in both circuits.*

Proof. By Proposition 4.3 it is equivalent to prove that all primitive elements of I_G satisfy the conditions (M1) and (M2) of Theorem 2.2 if and only if the circuits of the graph G satisfy the conditions (R1) through (R3).

For the forward direction we assume that all primitive elements of I_G satisfy the conditions (M1), (M2) of Theorem 2.2. Since all the circuits of G are also primitive elements, the conditions (R1), (R2) are followed. By Lemma 4.4 it follows the condition (R3).

For the other direction, we assume that every circuit of G satisfies (R1) through (R3). We will prove that every primitive element of I_G satisfies the conditions (M1) and (M2) of Theorem 2.2. Suppose not. Then, there exists a primitive walk w of I_G such that it has either an even chord or a bridge or it has two odd chords which cross effectively and they do not form an F_4. By Lemma 4.2 the walk w is either an even cycle or two odd cycles intersecting in exactly one vertex or $w = (c_1, w_1, c_2, w_2)$ where c_1, c_2 are odd vertex disjoint cycles and w_1, w_2 are walks which combine a vertex v_1 of c_1 and a vertex v_2 of c_2. If w has one of the first two forms, then the walk w is also a circuit, the hypothesis is contradicted. Therefore, the walk w is of the form $w = (c_1, w_1, c_2, w_2)$ where c_1, c_2 are odd vertex disjoint cycles and w_1, w_2 are walks which combine a vertex v_1 of c_1 and a vertex v_2 of c_2. Then the result follows with the same way as in the corresponding proof of the [3, Theorem 4.8]. □

5. Robustness and generalized robustness on toric ideals

In this section we study the robustness property and the generalized robustness property in the general case of toric ideals. In [3] Boocher et al. proved that all robust toric ideals which are generated by quadratics are graph ideals, see [3, Corollary 5.3]. As we can see in the next example, in the case of generalized robust toric ideals this is not true.

Example 5.1. We consider the set $A = \{(1,0,0,0,1), (0,1,1,1,1), (1,1,0,0,1), (0,0,1,1,1), (0,1,1,0,1), (1,0,0,1,1), (1,0,1,0,1), (0,1,0,1,1)\} \subseteq \mathbb{N}^5$. We compute by CoCoA, that the corresponding toric ideal is

$$I_A = \langle x_1 x_2 - x_3 x_4, x_1 x_2 - x_5 x_6, x_1 x_2 - x_7 x_8 \rangle,$$

for more see [7]. The ideal I_A is quadratic and there is not a graph G such that $I_A = I_G$, which means that I_A is not a graph ideal. Otherwise, there exists a graph G with three cycles of length four which have the edges x_1 and x_2 in common. Note that x_1, x_2 are vertex disjoint edges. This structure is impossible to happen for any simple graph G. Clearly for the toric ideal I_A we have that $Gr_A = \langle x_1 x_2 - x_3 x_4, x_1 x_2 - \,$

$x_5x_6, x_1x_2 - x_7x_8, x_3x_4 - x_5x_6, x_3x_4 - x_7x_8, x_5x_6 - x_7x_8 \rangle = \mathcal{M}_A$. By computations we check that $\mathcal{U}_A = \mathcal{M}_A$. It follows that the ideal I_A is generalized robust.

As we saw in Corollary 3.5 a robust graph ideal is also generalized robust. Next, we will see that this is a property not only for toric ideals of graphs, but for an arbitrary toric ideal as well. The main theorem of this section is that robust ideals are generated by indispensable binomials.

In [4, Theorem 2.12] Charalambous et al. described the indispensable elements of a toric ideal.

Theorem 5.2. *(See [4].) The ideal I_A is generated by indispensable binomials if and only if the Betti A-degrees $\mathbf{b}_1, \ldots, \mathbf{b}_k$ are minimal binomial A-degrees and $\beta_{0,\mathbf{b}_i} = 1, \forall\ i = 1, \ldots, k$.*

Next, we remind some useful definitions, as they are presented in [4], in order to understand the above theorem.

Let $A \subset \mathbb{Z}^n$ be a vector configuration so that $\mathbb{N}A$ is pointed and let I_A be its corresponding toric ideal. A vector $\mathbf{b} \in \mathbb{N}A$ is called a *Betti A-degree* if I_A has a minimal generating set containing an element of A-degree $\mathbf{b}$. We define the *A-graded Betti number* of I_A as the number of times that the vector $\mathbf{b}$ appears as the A-degree of a binomial in a given minimal generating set of the ideal. From [21] we know that the Betti A-degrees are independent of the choice of the minimal generating set of I_A.

Since the semigroup $\mathbb{N}A$ is pointed, we can partially order it with the relation:

$$\mathbf{c} \geq \mathbf{d} \iff \exists\ \mathbf{e} \in \mathbb{N}A : \mathbf{c} = \mathbf{d} + \mathbf{e}.$$

Also for $I_A \neq \{0\}$, the minimal elements of the set $\{\deg_A(\mathbf{x^u}) : \mathbf{x^u} - \mathbf{x^v} \in I_A\} \subset \mathbb{N}A$ with respect to $\geq$ are called *minimal binomial A-degrees*.

For any $\mathbf{b} \in \mathbb{N}A$ the following ideal is defined:

$$I_{A,\mathbf{b}} = (\mathbf{x^u} - \mathbf{x^v} \ : \ \deg_A(\mathbf{x^u}) = \deg_A(\mathbf{x^v}) \lneqq \mathbf{b}) \subset I_A.$$

Next, we define the graph $G(\mathbf{b})$. Based on this graph, another graph $(S_\mathbf{b})$ is defined, for more details see [4]. The following construction plays a key role in the proof of our main theorem.

Definition 5.3. For a vector $\mathbf{b} \in \mathbb{N}A$ we define the graph $G(\mathbf{b})$ to be the graph whose vertices are the elements of the fiber

$$\deg_A^{-1}(\mathbf{b}) = \{\mathbf{x^u} \ : \ \deg_A(\mathbf{x^u}) = \mathbf{b}\}$$

and on the edge set

$$E(G(\mathbf{b})) = \{\{\mathbf{x^u}, \mathbf{x^v}\} \ : \ \mathbf{x^u} - \mathbf{x^v} \in I_{A,\mathbf{b}}\}.$$

We consider the complete graph $S_\mathbf{b}$, whose vertices are the connected components $G(\mathbf{b})_i$ of $G(\mathbf{b})$. Let $T_\mathbf{b}$ be a spanning tree of $S_\mathbf{b}$. For every edge of $T_\mathbf{b}$ joining the components $G(\mathbf{b})_i$ and $G(\mathbf{b})_j$ of $G(\mathbf{b})$, we choose a binomial $\mathbf{x^u} - \mathbf{x^v}$ such that $\mathbf{x^u} \in G(\mathbf{b})_i$ and $\mathbf{x^v} \in G(\mathbf{b})_j$ correspondingly. Let $\mathcal{F}_{T_\mathbf{b}}$ the collection of these binomials.

Proposition 5.4. *(See [4, Proposition 2.2].) Let $\mathbf{b} \in \mathbb{N}A$. Every connected component of $G(\mathbf{b})$ is a complete subgraph. The graph $G(\mathbf{b})$ is not connected if and only if $\mathbf{b}$ is a Betti A-degree.*

The following corollary, which results from the above definitions, is useful for the proof of the main theorem of this section.

Corollary 5.5. *Let $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ be a minimal generator of I_A. The vertices $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{v}}$ of $G(\mathbf{b})$ belong to different connecting components of the graph $G(\mathbf{b})$.*

Proof. We assume that

$$I_A = \langle B_1 = \mathbf{x}^{\mathbf{u_1}} - \mathbf{x}^{\mathbf{v_1}}, \ldots, B_k = \mathbf{x}^{\mathbf{u_k}} - \mathbf{x}^{\mathbf{v_k}} \rangle,$$

where $\{B_1, \ldots, B_k\}$ is a minimal set of generators of I_A. We consider the minimal generator $B_1 = \mathbf{x}^{\mathbf{u_1}} - \mathbf{x}^{\mathbf{v_1}}$ and let $\mathbf{b} = \deg_A(\mathbf{x}^{\mathbf{u_1}} - \mathbf{x}^{\mathbf{v_1}})$ be the corresponding Betti A-degree. We assume that the vertices $\mathbf{x}^{\mathbf{u_1}}$ and $\mathbf{x}^{\mathbf{v_1}}$ belong to the same connecting component of the graph $G(\mathbf{b})$. By Proposition 5.4 we know that every connected component of $G(\mathbf{b})$ is a complete graph and therefore $\{\mathbf{x}^{\mathbf{u_1}}, \mathbf{x}^{\mathbf{v_1}}\}$ is an edge of $G(\mathbf{b})$. Thus, the binomial $\mathbf{x}^{\mathbf{u_1}} - \mathbf{x}^{\mathbf{v_1}} \in I_{A,\mathbf{b}}$, which means that $B_1 = \mathbf{x}^{\mathbf{u_1}} - \mathbf{x}^{\mathbf{v_1}} = \sum_j a_j M_j B_j$, where $j \in \{2, \ldots, k\}$. A contradiction arises since the set $\{B_1, \ldots, B_k\}$ is minimal. □

In [4] the authors proved the following theorems.

Theorem 5.6. *(See [4, Theorem 2.6].) The set $\mathcal{F} = \bigcup_{\mathbf{b} \in \mathbb{N}A} \mathcal{F}_{T_{\mathbf{b}}}$ is a minimal generating set for the ideal I_A.*

The converse of the above theorem is also true.

Theorem 5.7. *(See [4, Theorem 2.7].) Let $\mathcal{F} = \bigcup_{\mathbf{b} \in \mathbb{N}A} \mathcal{F}_{T_{\mathbf{b}}}$ be a minimal generating set of the ideal I_A. The binomials of $\mathcal{F}_{T_{\mathbf{b}}}$ determine a spanning tree $T_{\mathbf{b}}$ of $S_{\mathbf{b}}$.*

The next proposition will be used in the sequel.

Proposition 5.8. *(See [4, Proposition 2.4].) An A-degree $\mathbf{b}$ is minimal binomial A-degree if and only if every connected component of $G(\mathbf{b})$ is a singleton.*

Before presenting the main theorem, we put forward the following lemma, which was proved in [3] in the special case of toric ideals of graphs. We note that there is no difference in the general case of a toric ideal. By $\mu(I_A)$ we denote the number of minimal generators of the ideal.

Lemma 5.9. *Let I_A be a robust toric ideal. Then there is no term of an element of $\mathcal{U}_A$ which divides a term of another element of $\mathcal{U}_A$.*

Proof. Let I_A be a robust toric ideal. By definition, the set $\mathcal{U}_A$ is a minimal generating set for the ideal. Since the affine semigroup $\mathbb{N}A$ is pointed, the graded Nakayama Lemma applies that all minimal systems of generators of I_A have the same cardinality. Thus $|\mathcal{U}_A| = \mu(I_A)$. Also we remark that from the definition of robustness, it follows that $\mu(\text{in}_{\prec} I_A) = \mu(I_A)$, for all term orders $\prec$ and therefore

$$|\mathcal{U}_A| = \mu(\text{in}_{\prec} I_A), \text{ for all term orders } \prec.$$

We will prove the contrapositive. We suppose that $\mathcal{U}_A$ contains at least two binomials $f_1 = m_1 - m_2$ and $f_2 = n_1 - n_2$ such that the term m_1 divides the term n_1. Since the toric ideal I_A is prime, there exists a variable x which divides the monomial m_1 but not the monomial m_2. Taking $\prec$ to be the lex term order with x first, it follows that $(\text{in}_{\prec} f_1)|(\text{in}_{\prec} f_2)$. Therefore

$$\mu(\text{in}_{\prec} I_A) < |\mathcal{U}_A|,$$

a contradiction arises. □

We are ready now to continue with the proof of the main theorem of this article. In the following theorem we are proving that the robust toric ideals have a unique minimal system of generators.

Theorem 5.10. *Let I_A be a robust toric ideal. Then I_A is generated by indispensable binomials.*

Proof. Let I_A be a robust toric ideal. By definition the ideal I_A is minimally generated by its universal Gröbner basis. Let

$$M = \{B_1 = \mathbf{x^{u_1}} - \mathbf{x^{v_1}}, B_2 = \mathbf{x^{u_2}} - \mathbf{x^{v_2}}, \dots, B_k = \mathbf{x^{u_k}} - \mathbf{x^{v_k}}\}$$

be the universal Gröbner basis of I_A, which is also a minimal generating set for the ideal. We will prove that the binomials B_i are indispensable generators of I_A for every $i = 1, \dots, k$. By Theorem 5.2, it is enough to prove that the Betti A-degrees $\mathbf{b}_1, \dots, \mathbf{b}_k$ are minimal binomial A-degrees and $\beta_{0,\mathbf{b}_i} = 1, \forall\, i = 1, \dots, k$. As we saw before, we know that the Betti A-degrees are independent of the choice of the minimal generating set of I_A and therefore if we choose randomly one of them, it appears as a degree of an element of the set M as well.

Firstly, we prove that the Betti A-degrees are minimal. We consider the minimal generator $B_1 = \mathbf{x^{u_1}} - \mathbf{x^{v_1}}$ and let $\mathbf{b}_1 = \deg_A(\mathbf{x^{u_1}} - \mathbf{x^{v_1}})$ be its Betti A-degree. Let $G(\mathbf{b}_1)$ be the corresponding graph. By Proposition 5.8, the Betti A-degree $\mathbf{b}_1$ is minimal if and only if every connected component of $G(\mathbf{b}_1)$ is a singleton. Let $G(\mathbf{b}_1)_i$ be a connected component of $G(\mathbf{b}_1)$ which is not a singleton. Without loss of generality we can assume that the vertex $\mathbf{x^{u_1}}$ belongs to $G(\mathbf{b}_1)_i$ and let $\mathbf{x^w}$ be another vertex of $G(\mathbf{b}_1)_i$. Note that by Corollary 5.5, the vertex $\mathbf{x^{v_1}}$ does not belong in $G(\mathbf{b}_1)_i$ and let $G(\mathbf{b}_1)_j$ be its connected component. We consider the corresponding tree $T_{\mathbf{b}_1}$. Then, by Theorem 5.7, the edge $e_1 = \{\mathbf{x^{u_1}}, \mathbf{x^{v_1}}\}$ of the graph $S_{\mathbf{b}_1}$ is an edge of $T_{\mathbf{b}_1}$, where $G(\mathbf{b}_1)_i$ and $G(\mathbf{b}_1)_j$ are two of its vertices. We replace this edge by the edge $e = \{\mathbf{x^w}, \mathbf{x^{v_1}}\}$ in the tree $T_{\mathbf{b}_1}$. Obviously, we have a new spanning tree of the graph $S_{\mathbf{b}_1}$ and by Theorem 5.6 the set

$$M' = \{B_2 = \mathbf{x^{u_2}} - \mathbf{x^{v_2}}, \dots, B_k = \mathbf{x^{u_k}} - \mathbf{x^{v_k}}, B_{k+1} = \mathbf{x^w} - \mathbf{x^{v_1}}\}$$

is a minimal generating set of I_A. We remark that there is no element of M' which contains the term $\mathbf{x^{u_1}}$. Otherwise the monomial $\mathbf{x^{u_1}}$ appears at least twice in two different elements of the set M, which is the universal Gröbner basis of I_A, a contradiction arises by Lemma 5.9. We consider the binomial $B_1 = \mathbf{x^{u_1}} - \mathbf{x^{v_1}}$ which belongs to I_A. Since M' is a minimal generating set, the B_1 can be written as a linear combination of the elements of M'. We have that

$$\mathbf{x^{u_1}} - \mathbf{x^{v_1}} = \sum_{j=2}^{k+1} a_j m_j B_j, \text{ where } a_j \in \mathbb{K} \text{ and } m_j \text{ are monomials of } \mathbb{K}[\mathbf{x}].$$

As a result, there exists $j \in \{2, \dots, k, k+1\}$ such that $\mathbf{x^{u_j}} | \mathbf{x^{u_1}}$. Note that $j \neq k+1$, otherwise $\mathbf{x^w} | \mathbf{x^{u_1}}$ which is impossible, since $\mathbf{x^w} \neq \mathbf{x^{u_1}}$ and $\deg_A(\mathbf{x^w}) = \deg_A(\mathbf{x^{u_1}})$. Therefore, there is a term of an element of $M = \mathcal{U}_A$, which divides a term of another element of $\mathcal{U}_A$. By Lemma 5.9, we have a contradiction. So, we conclude that every connected component of $G(\mathbf{b}_1)$ is a singleton and therefore the Betti A-degree $\mathbf{b}_1$ is minimal.

It remains to prove that $\beta_{0,\mathbf{b}_i} = 1, \forall\, i = 1, \dots, k$. We assume that there exists $i \in \{1, \dots, k\}$ such that $\beta_{0,\mathbf{b}_i} \geq 2$. In other words, there are at least two elements of M with degree $\mathbf{b}_i$ and let them be $B_n = \mathbf{x^{u_n}} - \mathbf{x^{v_n}}$ and $B_m = \mathbf{x^{u_m}} - \mathbf{x^{v_m}}$. As we proved before, every connecting component of $G(\mathbf{b}_i)$ is a singleton, thus the graph $G(\mathbf{b}_i)$ has at least four connecting components; $\{\mathbf{x^{u_n}}\}$, $\{\mathbf{x^{v_n}}\}$, $\{\mathbf{x^{u_m}}\}$ and $\{\mathbf{x^{v_m}}\}$. Note that none of the above connecting components coincides with each other, otherwise we have a contradiction from Lemma 5.9. If we look at the corresponding tree $T_{\mathbf{b}_i}$, two of its edges are $e_1 = \{\mathbf{x^{u_n}}, \mathbf{x^{v_n}}\}$ and $e_2 = \{\mathbf{x^{u_m}}, \mathbf{x^{v_m}}\}$. Since a tree is a connected graph, then there exists a path which joins the edges

e_1, e_2. Therefore, at least one of the vertices $\{\mathbf{x^{u_n}}\}$, $\{\mathbf{x^{v_n}}\}$ appears in another one edge of $T_{\mathbf{b}_i}$, different from e_1. This means that at least one of the monomials $\mathbf{x^{u_n}}$ or $\mathbf{x^{v_n}}$ appears as a monomial term of another minimal generator of $M = \mathcal{U}_A$. A contradiction arises by Lemma 5.9. Thus $\beta_{0,\mathbf{b}_i} = 1$, $\forall\, i = 1, \ldots, k$ and the theorem follows. $\square$

The converse of the above theorem is not true, as we can see in the following remark.

Remark 5.11. The indispensability of the minimal generators of a toric ideal, is not a necessary condition for an ideal to be robust. For this claim we consider the following graph of Fig. 5.
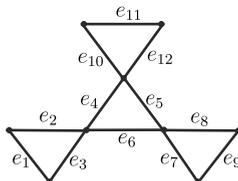

Fig. 5. An indispensable ideal which is not robust.

The corresponding toric ideal I_G has six minimal generators. These are:

$$B_1 = e_2 e_3 e_5 - e_1 e_4 e_6, \quad B_2 = e_4 e_5 e_{11} - e_6 e_{10} e_{12}, \quad B_3 = e_4 e_7 e_8 - e_5 e_6 e_9,$$

$$B_4 = e_2 e_3 e_{10} e_{12} - e_1 e_4^2 e_{11}, \quad B_5 = e_2 e_3 e_7 e_8 - e_1 e_6^2 e_9, \quad B_6 = e_7 e_8 e_{10} e_{12} - e_9 e_5^2 e_{11}.$$

By Corollary 3.7 all the above minimal generators of I_G are indispensable. We consider the walk $w = (e_2, e_1, e_3, e_6, e_7, e_9, e_8, e_5, e_{12}, e_{11}, e_{10}, e_4)$ of the graph G. By Theorem 2.1 the corresponding binomial $B_w = e_2 e_3 e_7 e_8 e_{10} e_{12} - e_1 e_4 e_5 e_6 e_9 e_{11}$ belongs to the Graver basis of I_G. We note that there are two sinks of w in distance one and therefore w is not strongly primitive. By Theorem 2.2 the binomial B_w is not minimal. Theorem 3.4 implies that the ideal I_G is not generalized robust. Therefore it is not robust.

We note that although the robust ideals are generated by indispensable binomials, this does not happen in the case of generalized robust ideals, see Example 3.6.

By Theorem 5.10 it follows the next corollary, in which we see that the property of robustness for a toric ideal, implies the generalized robustness property for it.

Corollary 5.12. *Let I_A be a toric ideal. If I_A is robust then it is generalized robust.*

Obviously, a necessary condition for the converse statement of the above corollary, is the uniqueness of the minimal system of generators of the toric ideal.

Corollary 5.13. *Let I_A be a toric ideal, such that it has a unique minimal system of generators. The ideal I_A is robust if and only if I_A is generalized robust.*

Remark 5.14. We note that the equality $M_A = Gr_A$ for robust toric ideals still remains an open problem, as it has been mentioned in [3]. We remark that the intersection of all minimal Markov bases M_A of a toric ideal, is called the indispensable subset of the universal Markov basis $\mathcal{M}_A$ and is denoted by S_A. Obviously, the following inclusions hold:

$$S_A \subseteq \mathcal{M}_A \subseteq Gr_A.$$

In [6] the authors gave a complete algebraic characterization for the elements of the sets S_A and $\mathcal{M}_A$. As it follows from the previous theorem, in the case of robust toric ideals we have that $S_A = \mathcal{M}_A$. An equivalent interesting question for robust toric ideals or generalized robust toric ideals is the equality $\mathcal{M}_A = Gr_A$. The significance of this problem is enhanced by the fact that toric ideals for which the universal Gröbner basis coincides with their Graver basis have important properties, as for example the equality between the Gröbner complexity and the Graver complexity of the ideal, see [13]. Also, in the case of robust toric ideals the following inclusions hold:

$$\mathcal{C}_A \subseteq (S_A = \mathcal{M}_A = M_A =) \, \mathcal{U}_A \subseteq Gr_A.$$

There are examples of families of ideals whose set of circuits is equal to the Graver basis. For example, Sturmfels proved this property for toric ideals defined by unimodular matrices, see [21, Proposition 8.11] and Villareal proved it for those defined by balanced matrices, see [16]. We know that toric ideals of graphs which are complete intersection are also circuit ideals, which means that every minimal generator of the ideal is a circuit, see [23, Theorem 5.1]. Thus, for those toric ideals we have that $\mathcal{C}_A = \mathcal{M}_A$. By Theorem 3.4, we get another family of ideals, i.e. robust toric ideals of graphs which are complete intersection, for which $\mathcal{C}_A = Gr_A$.

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