The structure of complete intersection graphs and their planarity
(joint work with A.Thoma)

ABSTRACT: Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subseteq \mathbb{N}^n$ be a vector configuration in $\mathbb{Q}^n$ and $\mathbb{N}A := \{l_1\mathbf{a}_1 + \cdots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}\}$ the corresponding affine semigroup, where $\mathbb{N}A$ is pointed, that is if $x \in \mathbb{N}A$ and $-x \in \mathbb{N}A$ then $x = \mathbf{0}$. We grade the polynomial ring $\mathbb{K}[x_1, \ldots, x_m]$ over an arbitrary field $\mathbb{K}$ by the semigroup $\mathbb{N}A$ setting $\deg_A(x_i) = \mathbf{a}_i$ for $i = 1, \ldots, m$. For $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$, we define the A-degree of the monomial $\mathbf{x}^\mathbf{u} := x_1^{u_1} \cdots x_m^{u_m}$ to be

$$\deg_A(\mathbf{x}^\mathbf{u}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A.$$

The toric ideal I_A associated to A is the prime ideal generated by all the binomials $\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v}$ such that $\deg_A(\mathbf{x}^\mathbf{u}) = \deg_A(\mathbf{x}^\mathbf{v})$.

Let G be a connected, undirected, finite, simple graph on the vertex set $V(G) = \{v_1, \ldots, v_n\}$. Let $E(G) = \{e_1, \ldots, e_m\}$ be the set of edges of G and $\mathbb{K}[e_1, \ldots, e_m]$ the polynomial ring in the m variables $e_1, \ldots, e_m$ over an arbitrary field $\mathbb{K}$. We will associate each edge $e = \{v_i, v_j\} \in E(G)$ with the element $a_e = v_i + v_j$ in the free abelian group $\mathbb{Z}^n$, with basis the set of the vertices of G, where $v_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the vector with 1 in the $i-$th coordinate of v_i. With I_G we denote the toric ideal I_{A_G} in $\mathbb{K}[e_1, \ldots, e_m]$, where $A_G = \{a_e \mid e \in E(G)\} \subset \mathbb{Z}^n$, known as graph ideals.

We study the complete intersection property of the graph ideals I_G. In general, the graph ideal I_G is complete intersection if and only if it can be generated by h binomials, where $h = m - n + 1$ if G is a bipartite graph or $h = m - n$ if G is not a bipartite graph. The answer is known in the case of bipartite graphs, i.e. graphs with no odd cycles. In the last years, several useful partial results have been proved and they provide key properties of complete intersection graph ideals .

We focus on the general case, where G is a random graph and we present a structural theorem which gives us necessary and sufficient conditions in which the toric ideal I_G is complete intersection. Moreover, we characterize the complete intersection graphs which are planar.