EACA´S Second International School
Computer Algebra and Applications
Valladolid, Spain

On the degree bounds of the Graver basis

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Joint work with A.Thoma

June 27, 2013

Basic Notations of Toric Ideals

Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_m\} \subseteq \mathbb{N}^n$ be a vector configuration and $\mathbb{N}A := \{l_1\mathbf{a}_1 + \cdots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}\}$ the corresponding affine semigroup. We grade the polynomial ring $\mathbb{K}[x_1, \dots, x_m]$ over an arbitrary field $\mathbb{K}$ by the semigroup $\mathbb{N}A$ setting $\deg_A(x_i) = \mathbf{a}_i$ for $i = 1, \dots, m$. For $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{N}^m$, we define the *$A$-degree* of the monomial $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \cdots x_m^{u_m}$ to be

$$u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A.$$

We denoted by $\deg_A(\mathbf{x}^{\mathbf{u}})$, while the usual degree $u_1 + \cdots + u_m$ of $\mathbf{x}^{\mathbf{u}}$ we denoted by $\deg(\mathbf{x}^{\mathbf{u}})$.

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We denoted by $\deg_A(\mathbf{x}^{\mathbf{u}})$, while the usual degree $u_1 + \cdots + u_m$ of $\mathbf{x}^{\mathbf{u}}$ we denoted by $\deg(\mathbf{x}^{\mathbf{u}})$.

Definition

The toric ideal I_A associated to A is the binomial ideal

$$I_A = < \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \ \deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}}) > .$$

A nonzero binomial $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ in I_A is called primitive if there exists no other binomial $\mathbf{x}^{\mathbf{w}} - \mathbf{x}^{\mathbf{z}}$ in I_A such that

$$\mathbf{x}^{\mathbf{w}} | \mathbf{x}^{\mathbf{u}} \text{ and } \mathbf{x}^{\mathbf{z}} | \mathbf{x}^{\mathbf{v}}.$$

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- The support of a monomial $x^{\mathbf{u}}$ of $\mathbb{K}[x_1, \ldots, x_m]$ is $supp(x^{\mathbf{u}}) := \{i \mid x_i \text{ divides } x^{\mathbf{u}}\}$ and the support of a binomial $B = x^{\mathbf{u}} - x^{\mathbf{v}}$ is $supp(B) := supp(x^{\mathbf{u}}) \cup supp(x^{\mathbf{v}})$.

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- An irreducible binomial B belonging to I_A is called a circuit of I_A if there is no binomial $B' \in I_A$ such that $supp(B') \subsetneqq supp(B)$.

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- The set of the circuits is denoted by $\mathcal{C}_A$ and it is a subset of the Graver basis.

Elements of Graph Theory

Elements of Graph Theory

Let G be a finite simple connected graph on the vertex set $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_q\}$ be the set of edges of G.

A walk of length s connecting $v_1 \in V(G)$ and $v_{s+1} \in V(G)$ is a finite sequence of the form

$$w = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_s, v_{s+1}\})$$

with each $\{v_j, v_{j+1}\} \in E(G)$. An even (respectively odd) walk is a walk of even (respectively odd) length. A walk $w = (e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, \ldots, e_q = \{v_s, v_{s+1}\})$ is called closed if $v_{s+1} = v_1$. A cycle is a closed walk

$$(\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_q, v_1\})$$

with $v_k \neq v_j$, for every $1 \leq k < j \leq q$.

Elements of Graph Theory

- We denote by $\mathbf{w}$ the subgraph of G with vertices the vertices of the walk and edges the edges of the walk w.
- A cut edge (respectively cut vertex) is an edge (respectively vertex) of the graph whose removal increases the number of connected components of the remaining subgraph.
- A graph is called biconnected if it is connected and does not contain a cut vertex. A block is a maximal biconnected subgraph of a given graph G.

Definition of I_G

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Definition

We denote by I_G the toric ideal I_{A_G} in $\mathbb{K}[e_1, \ldots, e_m]$, where $A_G = \{a_e \mid e \in E(G)\} \subset \mathbb{Z}^n$.

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Given an even closed walk $w = (e_1, \ldots, e_{2q-1}, e_{2q})$ of the graph G we denote by

$$E^+(w) = \prod_{k=1}^{q} e_{2k-1},\ E^-(w) = \prod_{k=1}^{q} e_{2k}$$

and by B_w the binomial

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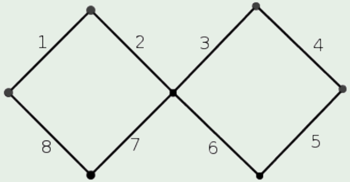
Theorem (Villarreal, 1995)

The toric ideal I_G is generated by binomials of this form

$$I_G = < B_w, \; w \text{ is an even closed walk of } G > .$$

Example



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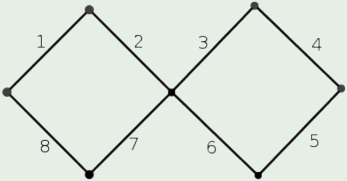

- $w_1 = (e_1, e_2, e_7, e_8)$ we have that $E^+(w_1) = e_1e_7$ and $E^-(w_1) = e_2e_8$ therefore $B_{w_1} = e_1e_7 - e_2e_8$.

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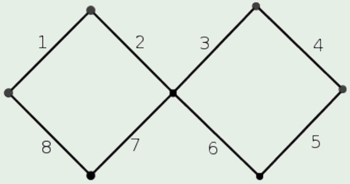

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- $w_2 = (e_3, e_4, e_5, e_6) \Longrightarrow B_{w_2} = e_3 e_5 - e_4 e_6$

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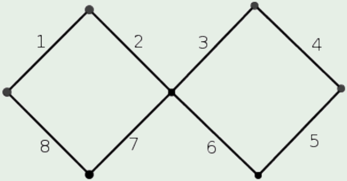

- $w_1 = (e_1, e_2, e_7, e_8)$ we have that $E^+(w_1) = e_1 e_7$ and $E^-(w_1) = e_2 e_8$ therefore $B_{w_1} = e_1 e_7 - e_2 e_8$.
- $w_2 = (e_3, e_4, e_5, e_6) \Longrightarrow B_{w_2} = e_3 e_5 - e_4 e_6$
- $w_3 = (e_1, e_2, \dots, e_8) \Longrightarrow B_{w_3} = e_1 e_3 e_5 e_7 - e_2 e_4 e_6 e_8$.

Therefore

$$B_{w_1}, B_{w_2}, B_{w_3} \in I_G.$$

The set of the circuits of I_G

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The following theorem determine the form of the circuits of a toric ideal of a graph G.

Theorem (Villareal, 1995)

Let G be a graph and let W be a connected subgraph of G. The subgraph W is the graph $\mathbf{w}$ of a walk w such that B_w is a circuit if and only if

1. W is an even cycle or
2. W consists of two odd cycles intersecting in exactly one vertex or
3. W consists of two vertex-disjoint odd cycles joined by a path.

The Graver basis of I_G

A walk w is primitive if and only if the corresponding binomial B_w is primitive.

Example

In the previous example, the binomial $B_{w_3} = e_1e_3e_5e_7 - e_2e_4e_6e_8$ is not primitive. There exists the even closed subwalk $w_1 = (e_1, e_2, e_7, e_8)$ of w_3, where its corresponding binomial is $B_{w_1} = e_1e_7 - e_2e_8$. We remark that $E^+(w_1)|E^+(w_3)$ and $E^-(w_1)|E^-(w_3)$.

The Graver basis of I_G

The next Theorem describes the form of the underlying graph of a primitive walk.

Theorem (Reyes, - ,Thoma, 2012)

Let G be a graph and let W be a connected subgraph of G. The subgraph W is the graph $\mathbf{w}$ of a primitive walk w if and only if

1. W is an even cycle or
2. W is not biconnected and
    1. every block of W is a cycle or a cut edge and
    2. every cut vertex of W belongs to exactly two blocks and separates the graph in two parts, the total number of edges of the cyclic blocks in each part is odd.

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It was conjectured that the degree of any element in the Graver basis Gr_A of a toric ideal I_A is bounded above by the maximal true degree of any circuit in $\mathcal{C}_A$.

True Circuit conjecture (Sturmfels, 1995)

Let us call t_A the maximal true degree of any circuit in $\mathcal{C}_A$. Then

$$\deg(B) \leq t_A,$$

for every $B \in Gr_A$.

Definition (Sturmfels)

Let $C \in \mathcal{C}_A$ be a circuit and regard its support $supp(C)$ as a subset of A. The index of the circuit C is denoted by index(C) and is defined as:

$$index(c) = [\mathbb{R}(supp(c)) \cap \mathbb{Z}\mathcal{A} : \mathbb{Z}(supp(c))].$$

The true degree of a circuit C is defined:

$$true\deg(C) = \deg(C) \cdot index(C)$$

Recall:

True Circuit conjecture

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There are several examples of families of toric ideals where the true circuit conjecture is true. It is also true for some families of toric ideals of graphs.

Recall:

True Circuit conjecture

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There are several examples of families of toric ideals where the true circuit conjecture is true. It is also true for some families of toric ideals of graphs.
However the true circuit conjecture is not true in the general case [- ,Thoma, 2011].

The Problem

To answer the conjecture we gave an infinite family of counterexamples by providing toric ideals of graphs such that:

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Problem

Does the degree of any element in the Graver basis Gr_A of a toric ideal I_A is bounded above by a constant times $(t_A)^2$ or a constant times $(t_A)^{2013}$?

Theorem

Let G be a graph and let C be a circuit in $\mathcal{C}_{A_G}$. Then

$$\textit{true}\deg(C) = \deg(C).$$

Theorem

Let G be a graph and let C be a circuit in $\mathcal{C}_{A_G}$. Then

$$\textit{true}\,\deg(C) = \deg(C).$$

Therefore to compare the maximum degrees of primitive elements of the I_G with t_A, its enough to compare the $\deg(B)$ with $\deg(C)$, for every $B \in Gr_A$ and for every $C \in \mathcal{C}_A$.

Let $n \geq 3$ be an odd integer. Let G_0^n be a cycle of length n. For $r \geq 0$ we define the graph G_r^n inductively on r. G_r^n is the graph taken from G_{r-1}^n by adding to each vertex of degree two of the graph G_{r-1}^n a cycle of length n.

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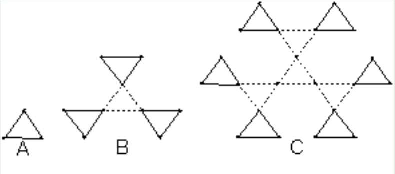
Example



We see in this figure the graphs $A = G_0^3$, $B = G_1^3$ and $C = G_2^3$.

We note that the graph G_r^n is Eulerian since by construction it is connected and every vertex has even degree.

Proposition

Let w_r^n be any even closed Eulerian walk of the graph G_r^n. The binomial $B_{w_r^n}$ is an element of the Graver basis of $I_{G_r^n}$ and

$$\deg(B_{w_r^n}) = \frac{1}{2}(n + n^2(\frac{(n-1)^r - 1}{n-2})).$$

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Proposition

Let $t_{A_{G_r^n}}$ the maximum degree of a circuit in the graph G_r^n. Then

$$t_{A_{G_r^n}} = n + (2r-1)(n-1).$$

Theorem

The degrees of the elements in the Graver basis of a toric ideal I_A cannot be bounded polynomially above by the maximal true degree of a circuit.

Or equivalently:
There is not a polynomial $f(x) \in \mathbb{R}[x]$, such that

$$\deg(B) \leq f(t_A), \forall B \in Gr_A, ,$$

for a toric ideal I_A, where t_A is the maximal true degree of the circuits $C \in \mathcal{C}_A$.

Thank you!!!