# RESEARCH SUMMARY

CHRISTOS TATAKIS

## 1. Previous Research (PhD Thesis)

1.1. **INTRODUCTION IN TORIC IDEALS.** Let $\mathfrak{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subseteq \mathbb{N}^n$ be a vector configuration in $\mathbb{Q}^n$ and $\mathbb{N}\mathfrak{A} := \{l_1\mathbf{a}_1 + \cdots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}\}$ the corresponding affine semigroup. Each vector $\mathbf{a}_i$ defines a monomial $\mathbf{t}^{\mathbf{a}_i}$ in the polynomial ring Laurent $\mathbb{K}[\mathbf{t}^{\pm 1}] := \mathbb{K}[t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}]$. We define the following homomorphism

$$\pi : \mathbb{N}^n \to \mathbb{Z}^d, \mathbf{u} = (u_1, \ldots, u_n) \mapsto \pi(\mathbf{u}) = u_1\mathbf{a_1} + \ldots + u_n\mathbf{a_n}.$$

The image of $\pi$, is the semigroup

$$\mathbb{N}\mathfrak{A} = \{\lambda_1\mathbf{a_1} + \ldots + \lambda_n\mathbf{a_n} : \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}.$$

The function $\pi$ arrases the homomorphism of semigroups:

$$\hat{\pi} : \mathbb{K}[\mathbf{x}] \to \mathbb{K}[\mathbf{t}^{\pm 1}], x_i \mapsto \mathbf{t}^{\mathbf{a_i}}.$$

**DEFINITION 1.1.** The toric ideal of $\mathfrak{A}$ is defined as the kernel of the $\hat{\pi}$ and is denoted by $I_{\mathfrak{A}}$.

We grade the polynomial ring $\mathbb{K}[x_1, \ldots, x_m]$ over any field $\mathbb{K}$ by the semigroup $\mathbb{N}\mathfrak{A}$ setting $\deg_{\mathfrak{A}}(x_i) = \mathbf{a}_i$ for $i = 1, \ldots, m$. For $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$, we define the $\mathfrak{A}$-*degree* of the monomial $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \cdots x_m^{u_m}$ to be

$$\deg_{\mathfrak{A}}(\mathbf{x}^{\mathbf{u}}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}\mathfrak{A}.$$

The *toric ideal* $I_{\mathfrak{A}}$ associated with $\mathfrak{A}$ is the prime ideal generated by all the binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ such that $\deg_{\mathfrak{A}}(\mathbf{x}^{\mathbf{u}}) = \deg_{\mathfrak{A}}(\mathbf{x}^{\mathbf{v}})$, see [39]. For such binomials, we define $\deg_{\mathfrak{A}}(\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}) := \deg_{\mathfrak{A}}(\mathbf{x}^{\mathbf{u}})$.

Toric ideals have a large number of applications in several areas including: algebraic statistics, biology, computer algebra, computer aided geometric design, dynamical systems, hypergeometric differential equations, integer programming, mirror symmetry, toric geometry and graph theory, see [1, 11, 13, 23, 39]. They are binomial ideals, i.e. polynomial ideals generated by binomials. There are several binomials in a toric ideal, such as minimal, indispensable, primitive, circuit and fundamental binomials which provide crucial information about the ideal and therefore they have been studied in more detail.

**DEFINITION 1.2.** A binomial $B \in I_{\mathfrak{A}}$ is called minimal if it belongs to at least one minimal system of generators of $I_{\mathfrak{A}}$.

The minimal binomials are finitely many, up to a scalar multiple. Their number is computed in [7] in terms of combinatorial invariants of a simplicial complex associated with the toric ideal. The minimal binomials cannot be written as a combination of binomials of smaller $\mathfrak{A}$-degrees, see [7, 33].

**DEFINITION 1.3.** A binomial $B \in I_{\mathfrak{A}}$ is called indispensable if there exists a nonzero constant multiple of it in every minimal system of binomial generators of $I_{\mathfrak{A}}$.

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A recent problem arising from Algebraic Statistics is to find when a toric ideal has a unique minimal system of binomial generators, see [6, 42]. To study this problem H. Ohsugi and T. Hibi introduced in [31] the notion of indispensable binomials and they gave necessary and sufficient conditions for toric ideals associated with certain finite graphs to possess unique minimal systems of binomial generators.

**DEFINITION 1.4.** An irreducible binomial $x^{\mathbf{u}^+} - x^{\mathbf{u}^-}$ in $I_{\mathfrak{A}}$ is called primitive if there exists no other binomial $x^{\mathbf{v}^+} - x^{\mathbf{v}^-} \in I_{\mathfrak{A}}$ such that $x^{\mathbf{v}^+}$ divides $x^{\mathbf{u}^+}$ and $x^{\mathbf{v}^-}$ divides $x^{\mathbf{u}^-}$. The set of the primitive elements of a toric ideal is called Graver basis and is denoted be $Gr_{\mathfrak{A}}$.

It follows from the definition that a non primitive binomial can be written as a sum of products of monomials times binomials of $I_{\mathfrak{A}}$ of smaller $\mathfrak{A}$-degrees therefore minimal binomials must be primitive, see also Lemma 3.1 of [29].
The support of a monomial $x^{\mathbf{u}}$ of $\mathbb{K}[x_1, \ldots, x_m]$ is $\text{supp}(x^{\mathbf{u}}) := \{i \mid x_i \text{ divides } x^{\mathbf{u}}\}$ and the support of a binomial $B = x^{\mathbf{u}} - x^{\mathbf{v}}$ is $\text{supp}(B) := \text{supp}(x^{\mathbf{u}}) \cup \text{supp}(x^{\mathbf{v}})$.

**DEFINITION 1.5.** An irreducible binomial $B$ belonging to $I_{\mathfrak{A}}$ is called a circuit of $I_{\mathfrak{A}}$ if there is no binomial $B' \in I_{\mathfrak{A}}$ such that $\text{supp}(B') \subsetneqq \text{supp}(B)$.

For a vector $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$ we define $\text{supp}(\mathbf{b}) = \{i | b_i \neq 0\}$. For a semigroup $\mathbb{N}\mathfrak{A}$ we denote the semigroup ring of $\mathbb{N}\mathfrak{A}$ by $\mathbb{K}[\mathbb{N}\mathfrak{A}]$. The semigroup ring $\mathbb{K}[\mathbb{N}\mathfrak{A}]$ is isomorphic to the quotient $\mathbb{K}[x_1, \ldots, x_m]/I_{\mathfrak{A}}$, see [23]. If $\mathcal{F}$ is a subset of $\{1, \ldots, n\}$, then $\mathfrak{A}_{\mathcal{F}}$ is the set $\{\mathbf{a}_i | \text{supp}(\mathbf{a}_i) \subset \mathcal{F}\}$. The semigroup ring $\mathbb{K}[\mathbb{N}\mathfrak{A}_{\mathcal{F}}]$ is a combinatorial pure subring of $\mathbb{K}[\mathbb{N}\mathfrak{A}]$, see [27] and for a generalization, see [26].

**DEFINITION 1.6.** A binomial $B \in I_{\mathfrak{A}}$ is fundamental if there exists a combinatorial pure subring $\mathbb{K}[\mathbb{N}\mathfrak{A}_{\mathcal{F}}]$ such that $\mathbb{K}[x_i | \mathbf{a}_i \in \mathfrak{A}_{\mathcal{F}}] \cap I_{\mathfrak{A}} = I_{\mathfrak{A}_{\mathcal{F}}} =< B >$.

These kinds of binomials are related with each other. The indispensable binomials are always minimal and the minimal binomials are always primitive. Also the fundamental binomials are circuits and indispensable, while the circuits are also primitive. An other relation between of them is given by Sturmfels in [39]. As we saw the set of primitive binomials forms the Graver basis of $I_{\mathfrak{A}}$ and is denoted by $Gr_{\mathfrak{A}}$. Also an irreducible binomial is called a circuit if it has minimal support. The set of circuits is denoted by $\mathcal{C}_{\mathfrak{A}}$. The relation among the set of circuits, the Graver basis and the universal Gröbner basis, which is denoted by $\mathcal{U}_{\mathfrak{A}}$, for a toric ideal $I_{\mathfrak{A}}$ is the following:

**THEOREM 1.7** (Sturmfels, [39])**.** *For any toric ideal* $I_{\mathfrak{A}}$ *we have* $\mathcal{C}_{\mathfrak{A}} \subset \mathcal{U}_{\mathfrak{A}} \subset Gr_{\mathfrak{A}}$.

From all the above information, we can see that the Graver basis is a very important set for a toric ideal. Also we remark that Graver basis has a very large interest for Algebraic Statistics since it is also a Markov basis for the ideal.

### 1.2. **TORIC IDEALS OF GRAPHS.**

In graph theory there are several monomial or binomial ideals associated with a graph depending on the properties one wishes to study, see [5, 9, 18, 19, 24, 36, 37, 41, 49, 50, 52]. One of them is the toric ideal of a graph which has been extensively studied over the last years, see [5, 9, 16, 15, 21, 22, 28, 29, 30, 31, 32, 48, 51, 50]. Toric ideals of graphs are the simplest toric ideals that we can use in order to understand the notions of circuits, fundamentals, primitive, minimal and indispensable binomials and how they are related. In the case of the toric ideal of a graph there are several articles in the literature that characterize these kinds of binomials, most of them for particular cases of graphs, see [16, 22, 29, 30, 31, 32, 50, 52].

Let $G$ be a connected finite graph on the vertex set $V(G) = \{v_1, \ldots, v_n\}$. Let $E(G) = \{e_1, \ldots, e_m\}$ be the set of edges of $G$ and $\mathbb{K}[e_1, \ldots, e_m]$ the polynomial ring in the $m$ variables $e_1, \ldots, e_m$ over a field $\mathbb{K}$. We will associate each edge $e = \{v_i, v_j\} \in E(G)$ with the element $a_e = v_i + v_j$ in the free abelian group $\mathbb{Z}^n$ with basis the set of vertices of $G$.

**DEFINITION 1.8.** We denote by $I_G$ the toric ideal $I_{A_G}$ in $\mathbb{K}[e_1, \ldots, e_m]$, where $A_G = \{a_e \mid e \in E(G)\} \subset \mathbb{Z}^n$.

A *walk* connecting $v_{i_1} \in V(G)$ and $v_{i_{q+1}} \in V(G)$ is a finite sequence of the form

$$w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_q}, v_{i_{q+1}}\})$$

with each $e_{i_j} = \{v_{i_j}, v_{i_{j+1}}\} \in E(G)$. We call a walk $w' = (e_{j_1}, \ldots, e_{j_t})$ a *subwalk* of $w$ if $e_{j_1} \cdots e_{j_t} | e_{i_1} \cdots e_{i_q}$. An edge $e = \{v, u\}$ of a walk $w$ may be denoted also by $(v, u)$ to emphasize the order that the vertices $v, u$ appear in the walk $w$. *Length* of the walk $w$ is called the number $q$ of edges of the walk. An even (respectively odd) walk is a walk of *even* (respectively odd) length. A walk $w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_q}, v_{i_{q+1}}\})$ is called *closed* if $v_{i_{q+1}} = v_{i_1}$. A *cycle* is a closed walk

$$(\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_q}, v_{i_1}\})$$

with $v_{i_k} \neq v_{i_j}$, for every $1 \leq k < j \leq q$. Depending on the property of the walk that we want to emphasize we may denote a walk $w$ by a sequence of vertices and edges $(v_{i_1}, e_{i_1}, v_{i_2}, \ldots, v_{i_q}, e_{i_q}, v_{i_{q+1}})$ or only vertices $(v_{i_1}, v_{i_2}, v_{i_3}, \ldots, v_{i_{q+1}})$ or only edges $(e_{i_1}, \ldots, e_{i_q})$ or the edges and vertices that we want to emphasize and sometimes we separate the walk into subwalks. For a walk $w = (e_{i_1}, \ldots, e_{i_s})$ we denote by $-w$ the walk $(e_{i_s}, \ldots, e_{i_1})$. Note that, although the graph $G$ has no multiple edges, the same edge $e$ may appear more than once in a walk. In this case $e$ is called *multiple edge of the walk* $w$. If $w'$ is a subwalk of $w$ then it follows from the definition of a subwalk that the multiplicity of an edge in $w'$ is less than or equal to the multiplicity of the same edge in $w$.

Given an even closed walk $w = (e_{i_1}, e_{i_2}, \ldots, e_{i_{2q}})$ of the graph $G$ we denote by

$$E^+(w) = \prod_{k=1}^{q} e_{i_{2k-1}}, \; E^-(w) = \prod_{k=1}^{q} e_{i_{2k}}$$

and by $B_w$ the binomial

$$B_w = \prod_{k=1}^{q} e_{i_{2k-1}} - \prod_{k=1}^{q} e_{i_{2k}}$$

belonging to the toric ideal $I_G$. Actually the toric ideal $I_G$ is generated by binomials of this form, see [50].

1.3. **RESULTS.** The purpose of my thesis was to study completely the toric ideal of a random graph $G$. The thesis is consisted of two articles. At the following we analyzed the main results of them.

- **E.Reyes, Ch.Tatakis, A.Thoma, Minimal generators of toric ideals of graphs, Adv. in Appl. Math., no. 1, 64-78, 48 (2012),** [34].

  This article has two research main parts. In the first one we describe completely in graph theoretically terms the Graver basis of $I_G$, for a random graph $G$. In the second one, we characterize with necessary and sufficient conditions minimal, indispensable and fundamental binomials of a toric ideal of a random graph. All of these characterizations as we will see in the next results are useful in solving problems at the theory of general toric ideals. For furthermore see [34, Theorems 3.2., 4.13., 4.14., 4.16.]. Also we give a theorem about when a toric ideal $I_G$ has a unique system of binomial generators.

- **Ch.Tatakis, A.Thoma, On the Universal Gröbner basis of toric ideals of graphs, J. Comb. Theory Series A 1540-1548, 118 (2011),** [44].

  This article has three parts. In the first one we use the characterizations of the above described article in order to determine the form of the binomials that belong to the universal Gröbner basis of the toric ideal of a graph, see [44, Theorem 3.4.]. Note that in general there are not a lot of kinds of toric ideals for which we know the universal Gröbner basis.

  It was conjectured that (True Circuit conjecture):

  **CONJECTURE 1.9.** [Sturmfels, 1995] The degree of any element in the Graver basis $Gr_A$ of a toric ideal $I_A$ is bounded above by the maximal true degree of any circuit in $\mathcal{C}_A$, [40, Conjecture 4.8], [19, Conjecture 2.2.10].

On the second part, the knowledge of the Graver basis and the set of the circuits of a toric ideal $I_G$, allows us to answer the true circuit conjecture by providing infinitely many counterexamples to it.

Finally, on the third part with the help of the knowledge of primitive elements of the $I_G$, we provide degree bounds for the binomials in the Graver basis and in the universal Gröbner basis of $I_G$. More explicitly in [10] J. De Loera, B. Sturmfels and R. Thomas proved that $d_n$ satisfies

$$n-2 \leq d_n \leq \binom{n}{2},$$

where $d_n$ is the largest degree of a binomial in the universal Gröbner basis for $I_{K_n}$. We are improving this result by proving that $d_n$ takes always the value $n-2$. At the end of the article we are proving for a random graph $G$ with $m$ vertices, where $m \geq 4$, that the largest degree $d$ of any binomial in the Graver basis (and in the universal Gröbner basis) for $I_G$ is $d \leq m-2$.

## 2. POSTDOC RESEARCH

Next, I continued to study problems arising from toric ideals of graphs. My research work developed further in the following articles.

- **Ch.Tatakis, Generalized robust toric ideals, J. of Pure and Applied Algebra, 263-277, 220 (1), 2016,** [43].

  An ideal $I$ is robust if its universal Gröbner basis is a minimal generating set for this ideal. In this paper, we generalize the meaning of robust ideals. An ideal is defined as generalized robust if its universal Gröbner basis is equal to its universal Markov basis.

  This article consists of two parts. In the first one, we study the generalized robustness on toric ideals of a graph $G$. We prove that a toric graph ideal is generalized robust if and only if its universal Markov basis is equal to the Graver basis of the ideal. Furthermore, we give a graph theoretical characterization of generalized robust graph ideals, which is based on terms of graph theoretical properties of the circuits of the graph $G$. In the second part, we go on to describe the general case of toric ideals, in which we prove that a robust toric ideal has a unique minimal system of generators, or in other words, all of its minimal generators are indispensable.

- **Ch.Tatakis, A.Thoma, Graver degrees are not polynomially bounded by true circuit degrees, J. of Pure and Applied Algebra, 2658-2665, 219, 2015,** [46].

  Let us call $t_A$ the maximal true degree of any circuit in $\mathcal{C}_A$. We recall the true circuit conjecture:

  $$\deg(B) \leq t_A,$$

  for every $B \in Gr_A$. As we saw the true circuit conjecture is not true in the general case. In [44] we gave an infinite family of counterexamples to the true circuit conjecture by providing toric ideals and elements of the Graver basis, the degrees of which are not bounded above by $t_A$. We note that in the counterexamples of [44] the degrees of the elements of the Graver basis were bounded by $t_A^2$. In this article we are proving that the degrees of the elements in the Graver basis of a toric ideal $I_A$ cannot be bounded polynomially above by the maximal true degree of a circuit. Or equivalently, the main theorem of this work is the following:

  There is not a polynomial $f(x) \in \mathbb{R}[x]$, such that

  $$\deg(B) \leq f(t_A), \forall B \in Gr_A,$$

  for a general toric ideal $I_A$.

- **Ch.Tatakis, A.Thoma, On complete intersection toric ideals of graphs, J. Algebraic Comb., 351-370, 38 (2) (2013),** [45]**.**

  The height of $I_G$ is either equal to $h = m - n + 1$ if $G$ is a bipartite graph, or equal to $h = m - n$ if $G$ is a non-bipartite graph, where $G$ is a connected graph, $m$ is the number of edges of $G$ and $n$ is the number of its vertices, see [50].

  **DEFINITION 2.1.** The toric ideal of $G$ is called a complete intersection if it can be generated by $h$ binomials.

  The complete intersection property of the toric ideals of graphs has been studied by a lot of authors, such as L. Doering and T. Gunston in [12], Simis [35], Katzman [22], Gitler, Reyes, and Villarreal see [14, 16]. However all of them studied the problem in the special case of bipartite graphs, i.e. graphs without odd cycles. In this article we study for a random graph $G$ the toric ideals $I_G$ which are complete intersections and we characterize all the complete intersection toric ideals of graphs which are normal, for furthermore see [45].

## 3. SUBMITTED PAPERS

- Ch.Tatakis, A.Thoma, On the relative size of toric bases, submitted.
- Y.Stamatiu, Ch.Tatakis, An algorithm which computes the universal Gröbner basis of graph ideals, submitted.

## 4. FUTURE RESEARCH/PROBLEMS

At the present moment my research progresses into two different paths. The first one continues to be relative to the toric ideals of graphs. More explicitly:

- **Ch.Tatakis, A.Thoma, The structure of complete intersection toric ideals of graphs and their planarity, in preparation.**

  As we saw before, the complete intersection property of the toric ideals of graphs was studied in the case of bipartite graphs by a lot of authors, such as L. Doering and T. Gunston in [12], Simis [35], Katzman [22], Gitler, Reyes, and Villarreal see [14, 16]. Gitler, Reyes, Villareal in [14] determined completely the form of the bipartite graphs for which the toric ideal $I_G$ is complete intersection see [14, 16]. They are the *ring graphs*. Given a graph $H$, we call a path $P$ an $H$-path if $P$ is non-trivial and meets $H$ exactly at its ends. A graph $G$ is a *ring graph* if each block of $G$ which is not an edge or a vertex can be constructed from a cycle by successively adding $H$-paths of length at least two that meet graphs $H$ already constructed in two adjacent vertices. It is a completely structural theorem of all complete intersection bipartite graphs. In the general case of random graphs the problem was studied by Tatakis, Thoma in [45] and by Bermejo, Garcia-Marco and Reyes in [3].

  Note also that the ideal of a general graph is much more complicated than the ideal of a bipartite graph. The generators of the toric ideal of a bipartite graph correspond with chordless even cycles of the graph. However the generators of the general graph have a more complicated structure, see [34].

  The aim of this work is to give a completely structural theorem of complete intersection toric ideals of random graphs. In order to accomplish this purpose we define two kinds of graphs, the notions of neckless and strip graphs. Our conviction is that all complete intersection toric ideal of graphs stems from these graphs, by a structural way.

  Finally, we study the complete intersection graphs which are planar. The problem of planarity in complete intersection graphs was firstly studied by Katzman in [22] and by Gitler, Reyes, Villareal in [16]. They proved that whenever a graph $G$ is a bipartite complete intersection, then it is planar. Next of them Bermejo, Garcia-Marco and Reyes provided an infinite family of non planar complete intersection graphs, see [3, Theorem 4.4].

  We characterize with a sufficient and necessary condition the complete intersection graphs which are planar, where $G$ is a random graph.

- **Ch.Tatakis, A.Thoma, M.Vladoiu, Complete intersection toric ideals of arbitrary large Markov complexity, in preparation.**

- **Y.Stamatiou, Ch.Tatakis, An algorithm which computes the universal Gröbner basis of graph ideals, in preparation,** [38]**.**
  In this paper we apply the theoretical knowledge of the Universal Gröbner basis of a toric ideal of a graph and the algorithmic knowledge of its Graver basis to give an algorithmic description of the Universal Gröbner basis of the toric ideal $I_G$. The algorithm decides in polynomial time whether an element of the Graver basis of $I_G$ belongs also in the $\mathcal{U}_A$. However, since the algorithm must test exhaustively all elements of the Graver basis for inclusion in the Universal Gröbner basis, the factor $|Gr_A|$ is present in the final complexity figure. For some classes of graphs this factor is not prohibitively large, i.e. it is polynomial in $n$. For instance, the class of graphs that consists of even cycles connected by paths. For such graphs, the complexity of the algorithm is $O(n^4)$ which is a polynomial in the number $n$ of vertices of the graph.

The second path moves on the gereral case of toric ideals, not special for graph ideals.

- **Ch.Tatakis, On the robustness of toric ideals, in preparation.**
  We call an ideal I in a polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ *robust*, if it can be minimally generated by its universal Gröbner basis. We note that the equality $M_A = Gr_A$ for robust toric ideals still remains an open problem, as it has been mentioned in [4]. We remark that the intersection of all minimal Markov bases $M_A$ of a toric ideal, is called the indispensable subset of the universal Markov basis $\mathcal{M}_A$ and is denoted by $S_A$. Obviously, the following inclusions hold:
  $$S_A \subseteq \mathcal{M}_A \subseteq Gr_A.$$
  In [8] the authors gave a complete algebraic characterization for the elements of the sets $S_A$ and $\mathcal{M}_A$. As it follows from the previous theorem, in the case of robust toric ideals we have that $S_A = \mathcal{M}_A$. An equivalent interesting question for robust toric ideals or generalized robust toric ideals is the equality $\mathcal{M}_A = Gr_A$. The significance of this problem is enhanced by the fact that toric ideals for which the universal Gröbner basis coincides with their Graver basis have important properties, as for example the equality between the Gröbner complexity and the Graver complexity of the ideal, see [17].
  Our aim is to answer the above question in the general case of the toric ideals.

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MITILINI, P.O. BOX 13, MITILINI (LESVOS) 81100, GREECE

*Email address*: chtataki@cc.uoi.gr