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Complete intersection toric ideals of graphs

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Basic Notations

Let G be a finite simple connected graph on the vertex set $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_q\}$ be the set of edges of G.

A walk of length s connecting $v_1 \in V(G)$ and $v_{s+1} \in V(G)$ is a finite sequence of the form

$$w = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_s, v_{s+1}\})$$

with each $\{v_j, v_{j+1}\} \in E(G)$. An even (respectively odd) walk is a walk of even (respectively odd) length. A walk $w = (e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, \ldots, e_q = \{v_s, v_{s+1}\})$ is called closed if $v_{s+1} = v_1$. A cycle is a closed walk

$$(\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_q, v_1\})$$

with $v_k \neq v_j$, for every $1 \leq k < j \leq q$.

Basic Notations

Given an even closed walk $w = (e_1, \ldots, e_{2q-1}, e_{2q})$ of the graph G we denote by

$$E^+(w) = \prod_{k=1}^{q} e_{2k-1},\ E^-(w) = \prod_{k=1}^{q} e_{2k}$$

and by B_w the binomial

$$B_w = E^+(w) - E^-(w)$$

belonging to the toric ideal I_G. The toric ideal I_G is in $\mathbb{K}[e_1, \ldots, e_n]$, where $\mathbb{K}$ is a field.

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Theorem (Villarreal 1995)

The toric ideal I_G is generated by binomials of this form

$$I_G = < B_w, \; w \text{ is an even closed walk of } G >$$

Example of I_G

- A set $\mathcal{A}$ of generators of I_G is minimal, if there is no other set $\mathcal{B}$ of generators of I_G such that $\mathcal{B} \subsetneq \mathcal{A}$.
- A generator of I_G is called minimal if it belongs in at least one minimal set of generators of I_G.

Let G be a graph on the vertex set $V(G) = \{v_1, \ldots, v_7\}$ and on the edge set $E(G) = \{e_1, \ldots, e_9\}$, where $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $e_3 = (v_3, v_1)$, $e_4 = (v_1, v_4)$, $e_5 = (v_4, v_5)$, $e_6 = (v_5, v_6)$, $e_7 = (v_6, v_7)$, $e_8 = (v_7, v_1)$ and $e_9 = (v_4, v_7)$. There are three generators of I_G:

$$B_{w_1} = e_1e_3e_9 - e_2e_4e_8, B_{w_2} = e_5e_7 - e_6e_9, B_{w_3} = e_1e_3e_5e_7 - e_2e_4e_6e_8.$$

Therefore

$$I_G =< B_{w_1}, B_{w_2}, B_{w_3} > .$$

We remark that B_{w_3} is not minimal. Thus $I_G =< B_{w_1}, B_{w_2} >$.

The Problem

The graph G is called bipartite if it does not contain an odd cycle. The height of I_G is equal to $h = q - n + 1$ if G is a bipartite graph or $h = q - n$ if G is not a bipartite graph, where q is the number of edges of the graph G and n the number of its vertices.

Definition

The toric ideal of G is called a complete intersection if it can be generated by h binomials.

We observe that saying that a graph is complete intersection, is exactly the same as saying that the corresponding I_G is complete intersection.

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Problem

When a toric ideal of a graph G is a complete intersection?

We knew the answer only in the case that G is a bipartite graph. (Simis 1998, Katzman 1999, Gitler-Reyes-Villarreal 2010).

Examples of C.I. graph / not C.I. graph

1. Let G be a graph on the vertex set $V(G) = \{v_1, \ldots, v_6\}$ and on the edge set $E(G) = \{e_1, \ldots, e_7\}$, where $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $e_3 = (v_3, v_4)$, $e_4 = (v_4, v_5)$, $e_5 = (v_5, v_6)$, $e_6 = (v_6, v_1)$ and $e_7 = (v_3, v_6)$. There are three generators of I_G:

   $$B_{w_1} = e_1e_7 - e_2e_6, B_{w_2} = e_4e_7 - e_3e_5, B_{w_3} = e_1e_3e_5 - e_2e_4e_6.$$

   Since $B_{w_3} = e_4B_{w_1} - e_1B_{w_2}$ only the binomials B_{w_1} and B_{w_2} are minimal. So the toric ideal I_G is generated from 2 binomials. Also h=q-n+1=7-6+1=2. Therefore I_G is a complete intersection toric ideal of the graph G.
2. We consider the above graph G in which we have replaced the edge e_7 by three other edges. Now all the three generators are minimal and therefore the corresponding I_G is not complete intersection.

Bipartite complete intersection graphs

- A graph is called planar, if it can be drawn in the plane so that its edges intersect only at their ends.
- A graph G has the primitive cycle property (PCP) if any two primitive cycles intersect in at most one edge.

For example the first graph at the previous example satisfies the PCP, but the second one it does not. Both graphs are planar.

Theorem (Katzman 1999)

Let G be a bipartite graph. The I_G is a complete intersection if and only if G is planar and satisfies the PCP.

The PCP is the reason that the first graph at the previous example is complete intersection and the second one it is not.

Bipartite complete intersection graphs

The next result is interesting because it shows how to construct all the complete intersection bipartite graphs.

- Given a graph H, we call a path P an H-path if P is non-trivial and meets H exactly in its ends.
- A graph G is a ring graph if each block of G which is not a bridge or a vertex, can be constructed from a cycle by successively adding H-paths of length at least 2 that meats graphs H already constructed in two adjacent vertices.

Theorem (Gitler,Reyes,Villarreal 2010)

Let G be a bipartite graph. The I_G is a complete intersection if and only if G is a ring graph.

For example the first graph at the previous example is a ring graph and therefore complete intersection, but the second one it is not.

Our aim is to characterize complete intersection graphs in the general case that G is a random graph.

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Theorem

Let G be a graph. Let H be a subset of $V(G)$ and G_H be the induced graph of H in G. The toric ideal I_G is complete intersection if and only if the toric ideal I_{G_H} is complete intersection for every H.

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Proposition

If the ideal I_G of a graph G is complete intersection then every cycle of G is contained in the w of at most one minimal generator B_w of I_G.

Example

Let G be a graph on the vertex set $V(G) = \{v_1, \ldots, v_7\}$ and on the edge set $E(G) = \{e_1, \ldots, e_9\}$, where the blocks of G, are the cycles $B_1 = \{e_1, e_2, e_3\}$, $B_2 = \{e_4, e_5, e_6\}$ and $B_3 = \{e_7, e_8, e_9\}$. The ideal I_G is not complete intersection since the cycle B_2 belongs in two minimal generators.

Contiguous blocks

Definition

Two blocks of a graph G are called contiguous it they are not cut edges and there is a path from the one to the other in which each edge of the path belongs to different block.

Theorem

Let G be a graph. If I_G is complete intersection then all blocks of G are bipartite except from at most two blocks. In the case that there are two non-bipartite blocks the two blocks are contiguous.

Example of Contiguous blocks

1. Let G be a graph on the vertex set $V(G) = \{v_1, \ldots, v_{11}\}$ and on the edge set $E(G) = \{e_1, \ldots, e_{14}\}$, where the blocks of G, are the cycles $B_1 = \{e_1, e_2, e_3\}$, $B_2 = \{e_4, e_5, e_6, e_7\}$, $B_3 = \{e_8, e_9, e_{10}, e_{11}\}$ and $B_4 = \{e_{12}, e_{13}, e_{14}\}$. The blocks B_1, B_4 are contiguous since they are cycles and there is a path $p = (e_7, e_8)$ from one to other in which $e_7 \in B_2$ and $e_8 \in B_3$ (there are a lot of paths from B_1 to B_4 but we need at least one with the above property).
2. Let G be a graph on the vertex set $V(G) = \{v_1, \ldots, v_8\}$ and on the edge set $E(G) = \{e_1, \ldots, e_{10}\}$, where the blocks of G, are the cycles $B_1 = \{e_1, e_2, e_3\}$, $B_2 = \{e_4, e_5, e_9, e_{10}\}$ and $B_3 = \{e_6, e_7, e_8, e_{11}\}$. The blocks B_1, B_3 are not contiguous since they are cycles and there is not a path from one to other in which each edge of the path belongs to different block.

Proposition

Let B be a complete intersection block of a graph G. All minimal generators are in the form B_w, where w is an even cycle.

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From this proposition and the previous theorem it follows the next theorem:

Theorem

Let G be a graph. If I_G is complete intersection then there exists at most one minimal generator B_w, such that w contains odd cycles of G. In this case B_w is a circuit whose two odd cycles belong to two different contiguous blocks.

The following result gives us a full description when a toric ideal I_G is complete intersection.

Theorem

Let G be a graph and let $B_1, \ldots, B_k$ be its blocks. I_G is complete intersection toric ideal if and only if
i) All generators except of at most one are in the form B_w where w is an even cycle and
ii) I_{B_i} is complete intersection toric ideal for all $1 \leq i \leq k$.

Example

For example, let G be the first in this presentation graph which we saw. We have computed that the $I_G = < B_{w_1}, B_{w_2} >$ where $B_{w_1} = e_1e_3e_9 - e_2e_4e_8$ and $B_{w_2} = e_5e_7 - e_6e_9$. There is a minimal generator B_w where w contains odd cycles. Its easy to see that every block of the graph is complete intersection. Therefore I_G is complete intersection.

A necessary and sufficient characterization of circuits was given by Villarreal:

Theorem (Villarreal 1995)

Let G be a graph. The binomial $B \in I_G$ is circuit if and only if $B = B_w$, where w is:

1. *an even cycle or*
2. *two odd cycles intersecting in exactly one vertex or*
3. *two vertex disjoint odd cycles joined by a path.*

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One property of complete intersection toric ideals of graphs, is that they are circuit ideals.

Theorem

Let G be a graph. If the toric ideal I_G is complete intersection then every minimal generator of I_G is a circuit.

References

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Thank you!!!